ON PRODUCTS OF GRAPHS AND REGULAR GROUPS

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ABSTRACT

A graph X is called a graphical regular representation (GRR) of a group $\mathscr{G}$ if the automorphism group of X is regular and isomorphic to $\mathscr{G}$. Watkins and Nowitz have shown that the direct product $\mathscr{G} \times \mathscr{H}$ of two finite groups $\mathscr{G}$ and $\mathscr{H}$ has a GRR if both factors have a GRR and if at least one factor is different from the cyclic group of order two. We give a new proof of this result, thereby removing the restriction to finite groups. We further show that the complement X' of a finite or infinite graph X is prime with respect to cartesian multiplication if X is composite and not one of six exceptional graphs.

A graph X is called a graphical regular representation (GRR) of a group $\mathscr{G}$, if the automorphism group $\mathscr{G}(X)$ of X is regular and isomorphic to $\mathscr{G}$. Watkins [6] has shown that the direct product $\mathscr{G} \times \mathscr{H}$ of two finite groups $\mathscr{G}$ and $\mathscr{H}$ has a GRR if each factor has a GRR and if both factors are different from the cyclic group $\mathscr{C}_2$ of order two. As $\mathscr{C}_2 \times \mathscr{C}_2$ has no GRR, one cannot let both factors be isomorphic to $\mathscr{C}_2$, but if only one factor is a $\mathscr{C}_2$ the theorem still holds (Watkins and Nowitz [5]). The aim of this paper is to remove the restriction to finite groups. The theorem that $\mathscr{G} \times \mathscr{C}_2$ has a GRR if the finite or infinite group $\mathscr{G}$ is different from $\mathscr{C}_2$ and has a GRR is an immediate consequence of the following: If the finite or infinite graph X is composite with respect to cartesian multiplication its complement X' is prime, unless X is one of six exceptional graphs.

We only consider simple graphs. The vertex set of a graph X will be denoted by $V(X)$ and the set of edges by $E(X)$. We consider the edges to be unordered pairs $[a,b]$ of different vertices of X. With $|X|$ we denote the number of vertices in X and with $|E(X)|$ the number of edges. Further, we will use the symbol $\square$ for the empty graph and C_n for the complete graph on n vertices. The complement C_n' of

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C_n is called the totally disconnected graph D_n. C_1 is called the trivial graph. We say X is nontrivial if it has at least two vertices.

A permutation ζ of $V(X)$ is an automorphism of X, if $[\zeta x, \zeta y] \in E(X)$ if and only if $[x, y] \in E(X)$. The automorphism group of X will be denoted by $\mathcal{G}(X)$. If there exists exactly one automorphism to every pair $x, y \in V(X)$ mapping x into y, $\mathcal{G}(X)$ is called regular.

Let A be a subset of $V(X)$. We define a graph $S(A)$ on A by connecting two vertices in A if and only if they are connected in X. $S(A)$ is said to be spanned by A. We will only consider such subgraphs of X which are spanned by some subset of $V(X)$.

If X and Y are graphs $X - Y$ is defined on the graph spanned by $V(X) - V(Y)$. A subgraph Y of X is called externally related [1] if every vertex of $X - Y$ is either connected with all vertices of Y or with none.

The union $X \cup Y$ of two graphs is defined by $V(X \cup Y) = V(X) \cup V(Y)$ and $E(X \cup Y) = E(X) \cup E(Y)$. Analogously, we define $X \cap Y$. It is easily seen that the union of externally related subgraphs with nonempty intersection is externally related and that their intersection is externally related.

The automorphism group of a graph is not regular, if X has a nontrivial proper subgraph Y with nontrivial automorphism group.

The cartesian product $X \times Y$ of the graphs X and Y is defined on the cartesian product $V(X) \times V(Y)$ of their sets of vertices by $[(x_1, y_1), (x_2, y_2)] \in E(X \times Y)$ if $x_1 = x_2$ and $[y_1, y_2] \in E(Y)$, or if $[x_1, x_2] \in E(X)$ and $y_1 = y_2$. The projection p_X maps every vertex $(x, y) \in V(X \times Y)$ into x and p_Y maps (x, y) into y. If $(a, b) \in V(X \times Y)$ we define the X-layer $X^{(a,b)}$ of $X \times Y$ as the subgraph of $V(X \times Y)$ spanned by the set $\{(x, b) \mid x \in V(X)\}$. Analogously, we define $Y^{(a,b)}$. Clearly p_X maps $X^{(a,b)}$ isomorphically onto X as p_Y maps the layer $Y^{(a,b)}$ onto Y.

Cartesian multiplication is an associative, commutative operation with unity, namely the trivial graph. It is distributive with respect to the sum of graphs, if we define the sum $X + Y$ as the union of disjoint graphs.

A graph is called composite with respect to cartesian multiplication if it is the product of nontrivial graphs. A nontrivial graph which is not composite is prime. In [2] it has been shown that the product $X \times Y$ of finite or infinite connected prime graphs has a regular automorphism group if and only if X and Y are nonisomorphic and have regular automorphism groups.

THEOREM 1. *The only composite graphs in the class of all finite and infinite* *graphs whose complement is not prime with respect to cartesian multiplication are* $C_2 \times C_2$, $C_2 \times D_2$; $C_2 \times C_2 \times C_2$, $C_2 \times C_4$; $C_3 \times C_3$ and $C_2 \times K$, *where* K *arises from* C_4 *by deletion of an edge.*

PROOF. Let ζ be an isomorphism from $(A \times B)'$ to $C \times D$, where A, B, C, D are nontrivial graphs. We consider the case first, when at least one factor, say A, has more than four vertices. Then one factor of $C \times D$, say C, has at least three vertices. If B consists only of two vertices u, v we consider $\zeta^{-1} C^g$ for a C-layer C^g of $C \times D$. There is an A-layer in $A \times B$ with which $\zeta^{-1} C^g$ has at least two vertices in common. Suppose (x,u) and (y,u) are in $\zeta^{-1} C^g$. In $(A \times B)'$ the vertex (z,v) is connected with (x,u), (y,u) for any z in A different from x and y. Thus, $\zeta(z,v)$ is in C^g for any such z. Let r, s be two vertices in A different from x, y and z. Then (z,u) is connected with (r, v) and (s, v) in $(A \times B)'$, which implies that $\zeta\,(z, u)$ is also in C^g. Finally, (x,v) is connected in $(A \times B)'$ with (y,u), (z,u), and (y,v) with (x,u), (z,u) for any z in A different from x and y. Thus, (x,v) and (y,v) are also in C^g and D is trivial.

Now let B have at least three vertices. We choose a vertex u in B and two arbitrary others, say v and w. Further let x, y, z and t be four vertices in A. Then the vertices (x,u), (z,v), (t,w) form a triangle and are mapped by ζ into a layer of $C \times D$ with respect to one of the factors. Let ζ map this triangle into the layer C^h. As (y,u) is connected with (z,v) and (t,w) in $(A \times B)'$, $\zeta(y,u)$ also is in C^h. As before, one concludes now that ζ maps $A^{(x,u)}$ and $A^{(x,v)}$ into C^h. As v was arbitrary, ζ maps every A-layer of $A \times B$, and hence $A \times B$, into C^h, which means that D is trivial.

We can restrict ourselves to the case now where all factors have at most four vertices. The number of edges in $A \times B$ added to the number of edges in $C \times D$ is the number of edges in the complete graph on $|A \times B|$ vertices. Denoting the number of vertices in A, B, C, D with a, b, c, d we therefore have:

$$b\cdot|E(A)| + a\cdot|E(B)| + d\cdot|E(C)| + c\cdot|E(D)| = \tfrac{1}{2}ab(ab-1). \tag{1}$$

Using the fact that $|E(X)| \leqq \frac{1}{2} \cdot x(x-1)$ for any graph X on x vertices we arrive at the following inequality:

$$a + b + c + d \geqq ab + 3. \tag{2}$$

We can assume without loss of generality that

$$4 \geqq a \geqq c \geqq d \geqq b \geqq 2. \tag{3}$$

Thus $4a \geqq ab + 3$, which implies $2 \leqq b \leqq 3$. If $b = 3$ we have $c + d \geqq 2a$ by (2) and $2a \geqq c + d$ by (3). Therefore, $a = c = d$, and since $ab = cd$ all constants are equal to three. In this case equality holds in (2) and A, B, C, D have to be complete. It is readily verified that $C_3 \times C_3$ is indeed isomorphic to its complement.

If $b = 2$ Eq. (2) gives $c + d \geqq a + 1$, wherefrom follows $2c \geqq a + 1$, $4c \geqq 2a + 2 = cd + 2$ and $c(4 - d) \geqq 2$. Thus $2 \leqq d \leqq 3$.

Now let $b = 2$ and $d = 3$. Then (3) implies $4 \geqq c \geqq 3$. By $2a = 3c$ the number c has to be even. So c cannot be three. But it cannot be four either, because then a would be equal to six.

If b and d are two, we have $a = c$ by $ab = cd$. If both graphs B and D do not have an edge (1) cannot be satisfied. If one of these graphs has an edge, but not the other, (1) can only be satisfied if all the other graphs are complete. Let $B \cong D_2$. Then we have

$$(C_a \times D_2)' \cong C_a \times C_2,$$

which can only hold if $a = 2$. For if $a > 2$ the graph $C_a \times C_2$ would contain a triangle, but $(C_a \times D_2)'$ does not contain any triangles. This gives the solutions $C_2 \times C_2$ and $C_2 \times D_2$. Let $B \cong D \cong C_2$. Then (1) implies

$$|E(A)| + |E(C)| = a(a - \tfrac{3}{2}). \tag{4}$$

If $a = 4$ the graphs A and C have at most six edges. By (4) one of these graphs, say C has to contain at least five edges. If C has five edges A also has five edges, and both graphs arise from C_4 by deletion of an edge. We call this graph K, and get the solution

$$(K \times C_2)' \cong K \times C_2.$$

If C has six edges it is complete and A has only four edges. There are no triangles in $(C \times D)'$ and henceforth none in $A \times B$ or A. Thus A is a quadrangle $C_2 \times C_2$ and one readily verifies

$$((C_2 \times C_2) \times C_2)' \cong C_4 \times C_2.$$

For $a = 3$ Eq. (4) cannot be satisfied because a is odd. The case $a = 2$ gives $A \cong D_2$, $C \cong C_2$ or $A \cong C_2$, $C \cong D_2$, a solution we already have.

THEOREM 2. *If the finite or infinite group $\mathcal{G}$ has a GRR and if $\mathcal{G} \not\cong \mathcal{C}_2$, then the group $\mathcal{G} \times \mathcal{C}_2$ also has a GRR.*

PROOF. Let X be a *GRR* of $\mathcal{G}$. Since X and X' have the same group and none of the exceptional graphs in Theorem 1 has a regular group we can assume X to be prime. As $\mathcal{G} \neq \mathcal{C}_2$ the graphs X and C_2 are not isomorphic. Thus, $X \times C_2$ is a GRR of $\mathcal{G} \times \mathcal{C}_2$.

The preceding proof is based on the same idea as the proof of Watkins and Nowitz [5] for the finite case. Where the cited proof uses the observation that the complement X' of a finite graph X is relatively prime to C_2 if X is a GRR and admits a C_2 as a factor, we use the stronger result of Theorem 1.

We remark that $\mathcal{C}_2^n$ has a GRR for all $n \neq 2,3,4$, as has been proved in [4]. In [3] it has been remarked that the weak product $\mathcal{C}_2^\omega$ also has a GRR for any infinite cardinal ω.

Definition. [6] The product $X \cdot Y$ of the graphs X and Y is defined on $V(X) \times V(Y)$ by $[(x_1,y_1),(x_2,y_2)] \in E(X \cdot Y)$ if $x_1 = x_2$ and $[y_1,y_2] \in E(Y)$, or if $[x_1,x_2] \in E(X)$ and $y_1 \neq y_2$.

All the Y-layers of $X \cdot Y$ are mapped isomorphically into Y by the projection p_y and all X-layers are totally disconnected. If b is a vertex not in Y^a it is either disconnected with all vertices of Y^a or connected with all but one. In general this product is neither commutative nor associative.

Theorem 3. *$\mathcal{G}(X \cdot Y) = \mathcal{G}(X) \times \mathcal{G}(Y)$ if X and Y have more than two vertices and regular groups.*

Proof. Let ζ be an automorphism of $X \cdot Y$ and Y^a a Y-layer of $X \cdot Y$. Consider the graph Z spanned by the projection $p_X V(\zeta Y^a)$ of $V(\zeta Y^a)$ into X. If Z is not externally related in X, there exist vertices $x \in V(X - Z)$ and $s,t \in V(Z)$ such that x is connected with s, but not with t. Let (s,u) be a vertex in ζY^a and v a vertex in Y different from u. Then (x,v) is connected with all but one of the vertices in ζY^a, and this one has to be a vertex whose projection into X is t. Let (t,w) be this vertex. Obviously, it is the only vertex in $Y^{(t,w)}$ which is in ζY^a. If (p,q) is any vertex in ζY^a with $p \neq t$ it has to be connected with (x,v) and therefore $[x,p] \in E(X)$. If $q \neq u$ the vertex (x,q) is connected with (s,u) but not with (t,w) and (p,q), which is not possible because the last three vertices are in ζY^a. Thus, ζY^a consists of the vertex (t,w) and the vertices (r,u) with $r \in V(Z) - \{t\}$. This means that all vertices of ζY^a except possibly (t,w) are in the layer $X^{(s,u)}$ and ζY^a is totally disconnected or all edges are incident with (t,w), in contradiction to the regularity of $\mathcal{G}(Y)$. Thus Z is externally related.

A regular graph on at least three vertices is connected. Therefore, there has to exist a vertex x in $X - Z$ which is connected with a vertex s in Z if $X \neq Z$. If Z

contains at least one additional vertex t besides s we show that it is not possible that Y^a is equal to $Z \cdot Y$. Suppose ζY^a is equal to $Z \cdot Y$ and let u, v be any two vertices in Y. Then (x, v) is connected with (s, u) but not with (s, v) and (t, v), which is not possible, because (x, v) has to be connected with all but one of the vertices in $\zeta Y^a = Z \cdot Y$.

If $Z = X$ it is also not possible that $\zeta Y^a = Z \cdot Y$, since ζY^a is a proper subgraph of $X \cdot Y$ and cannot be mapped onto $X \cdot Y$ by the automorphism ζ.

In case Z is not complete, there are two nonadjacent vertices, say s, t in Z. We want to show that at least one of the layers $Y^{(s,u)}$ and $Y^{(t,u)}$ for $u \in Y$ contains a vertex which is not in ζY^a. In order to do that, we take a vertex (r, v) in $Z \cdot Y - \zeta Y^a$. As $Y^{(r,v)}$ is connected and $Y^{(r,v)} \cap \zeta Y^a \neq \square$, we can assume that (r, v) is adjacent to a vertex in $Y^{(r,v)} \cap \zeta Y^a$, and hence with all but one of the vertices in ζY^a. If r is equal to s or t we are through, if not, we observe that (r, v) is not adjacent to (s, v) and (t, v); and therefore, at least one of these vertices, say (s, v) has to be in $Z \cdot Y - \zeta Y^a$.

As $Y^{(s,u)} \neq Y^{(s,u)} \cap \zeta Y^a \neq \square$, we can find by the connectedness of $Y^{(s,u)}$ a vertex b in $Y^{(s,u)} - \zeta Y^a$ which is adjacent to a vertex c in $Y^{(s,u)} \cap \zeta Y^a$. The vertex b is therefore connected with all vertices in ζY^a but one, say d. Since there are no edges between $Y^{(s,u)}$ and $Y^{(t,u)}$ the vertex d has to be in $Y^{(t,u)}$, and it is the only vertex of ζY^a in this Y-layer. Clearly, s is adjacent to any vertex z in Z other than s or t. Let e be a neighbor of d in $Y^{(t,u)}$. Then e is not in ζY^a and adjacent to all but one of the vertices of ζY^a. This vertex must be c, and clearly t is adjacent to any vertex z in Z other than s or t. The transposition of s and t is therefore a nontrivial automorphism of Z which fixes at least one vertex of Z. Since Z is externally related, it is also an automorphism of X, in contradiction to the regularity of $\mathcal{G}(X)$.

We have thus shown that Z is complete, and as Z is externally related and $\mathcal{G}(X)$ regular, Z has to be trivial. This means that ζ maps every Y-layer into a Y-layer. The mapping ζ^{-1} is also an automorphism and has the same property; hence, ζ maps every Y-layer onto a Y-layer.

To complete the proof, it remains to be shown that ζ also maps every X-layer onto an X-layer. By the connectedness of X, it suffices to show that for arbitrary $u \in Y$ $p_y\zeta(t, u) = p_y\zeta(s, u)$, if $[s, t] \in E(X)$. If $[s, t] \in E(X)$ the vertex (s, u) is connected with all vertices of $Y^{(t,u)}$ except (t, u). Therefore (s, u) is connected with all vertices of

$$\zeta Y^{(t,u)} = Y^{\zeta(t,u)}$$

except $\zeta(t,u)$. Thus $\zeta(s,u)$ and $\zeta(t,u)$ have the same Y-components.

THEOREM 4. *Let $\mathscr{G}$ and $\mathscr{H}$ be finite or infinite groups with GRR. If not both are isomorphic to $\mathscr{C}_2$ the group $\mathscr{G} \times \mathscr{H}$ also has a GRR.*

PROOF. If $\mathscr{G}$ and $\mathscr{H}$ are different from $\mathscr{C}_2$ this is an immediate consequence of Theorem 3 as the direct product of regular groups is regular. If one of the groups is a $\mathscr{C}_2$ we get a *GRR* of $\mathscr{G} \times \mathscr{H}$ by Theorem 2.

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